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On a Class of Differential Equations.

By Alexander Chessin.

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1. Let us denote by $D^{(n)}$ the operation

$$A_0\frac{d^n}{dx^n}+A_1\frac{d^{n-1}}{dx^{n-1}}+\ldots.+A_{n-1}\frac{d}{dx}+A_n,$$

where the coefficients $A_0, A_1, \ldots., A_n$ are functions of x, and consider a function y of x defined by the linear differential equation of order mn,

$$y_m+a_1y_{m-1}+a_2y_{m-2}+\ldots.+a_my=f(x), \tag{1}$$

with the constant coefficients $a_1, a_2, \ldots., a_m$, and where

$$y_k=D^{(n)}y_{k-1};\quad y_0=y;\qquad k=1, 2, \ldots., m. \tag{2}$$

The integration of (1) can be reduced to the integration of an equation of the form

$$D^{(n)}y=f(x), \tag{3}$$

2. To show this, let us make the substitution

$$y_1=-\lambda y+z, \tag{4}$$

from which it follows that

$$y_k=-\lambda y_{k-1}+z_{k-1};\qquad k=1, 2, \ldots., m; \tag{4 bis}$$

the functions $z_1, z_2, \ldots.$ being defined by equations similar to (2), viz.

$$z_k=D^{(n)}z_{k-1};\quad z_0=z;\qquad k=1, 2, \ldots., m-1.$$

Then equation (1) becomes

$$z_{m-1} + (a_1 - \lambda) z_{m-2} + (a_2 - a_1\lambda + \lambda^2) z_{m-3} + \ldots\ldots \\ + [a_{m-1} - a_{m-2}\lambda + a_{m-3}\lambda^2 - \ldots\ldots + (-1)^{m-1}\lambda^{m-1}] z \\ + [a_m - a_{m-1}\lambda + a_{m-2}\lambda^2 - \ldots\ldots + (-1)^m\lambda^m y] = f(x),$$

and, by giving to λ the value of any one of the roots of the equation

$$a_m - a_{m-1}\lambda + a_{m-2}\lambda^2 - \ldots\ldots + (-1)^m\lambda^m = 0, \tag{5}$$

we arrive at an equation similar to (1) but of order $(m - 1)n$, viz.

$$z_{m-1} + b_1 z_{m-2} + b_2 z_{m-3} + \ldots\ldots + b_{m-1} z = f(x), \tag{6}$$

where $$b_k = a_k - a_{k-1}\lambda + a_{k-2}\lambda^2 - \ldots\ldots + (-1)^k\lambda^k. \tag{7}$$

3. Now, suppose $[z]$ to be the general solution of (6). It will contain $(m - 1)n$ arbitrary constants. To obtain the function y, we only need to substitute $[z]$ for z into (4) and integrate the resulting equation

$$y_1 = -\lambda y + [z] \tag{8}$$

which is, obviously, of the form (3). This integration will introduce n new arbitrary constants, bringing the total number of them to mn. The integral so obtained is, therefore, the general solution of the given differential equation.

On the other hand, the general solution of (6) may be obtained by a series of successive and similar operations. Thus, the substitution

$$z_1 = -\mu z + u, \tag{9}$$

with the condition that μ be a root of the equation

$$b_{m-1} - b_{m-2}\mu + b_{m-3}\mu^2 - \ldots\ldots + (-1)^{m-1}\mu^{m-1} = 0 \tag{10}$$

reduces the integration of (6) to the integration of the differential equation of order $(m - 2)n$,

$$u_{m-2} + c_1 u_{m-3} + c_2 u_{m-4} + \ldots\ldots + c_{m-2} u = f(x), \tag{11}$$

where $$c_k = b_k - b_{k-1}\mu + b_{k-2}\mu^2 - \ldots\ldots + (-1)^k\mu^k, \tag{12}$$

and $[u]$ being the general solution of (11), the general solution of (6) is obtained by the integration of the equation

$$z_1 = -\mu z + [u]$$

which, again, is of the form (3).

By means of these successive reductions, we will finally arrive at an equation of the form

$$w_1 + \theta w = f(x), \tag{13}$$

i. e. of the form (3). Q. E. D.

4. In practice, however, it would not be convenient to proceed in this manner. We will show how the single integration of the equation (13) at once furnishes the general solution of (1).

To begin with, it can be shown that equation (13) holds good for all the values of θ which satisfy the equation $F(\theta) = 0$, where

$$F(\theta) \equiv a_m - a_{m-1}\theta + a_{m-2}\theta^2 - \dots\dots + (-1)^m\theta^m.$$

In fact, the values of θ must satisfy the equation resulting from the elimination of $\lambda, \mu, \nu, \xi, \dots\dots$ between the equations (5), (10) and those formed successively in a similar manner, viz.

$$\left.\begin{array}{l} c_{m-2} - c_{m-3}\nu + c_{m-4}\nu^2 - \dots\dots + (-1)^{m-2}\nu^{m-2} = 0, \\ d_{m-3} - d_{m-4}\xi + d_{m-5}\xi^2 - \dots\dots + (-1)^{m-3}\xi^{m-3} = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{array}\right\} \tag{14}$$

where $d_k = c_k - c_{k-1}\nu + c_{k-2}\nu^2 - \dots\dots + (-1)^k\nu^k$, etc. Now, the form of the coefficients in these equations enables us to effect the elimination successively in a very simple manner. In general, let

$$\begin{aligned} \Phi(\eta) &= \alpha_p - \alpha_{p-1}\eta + \alpha_{p-2}\eta^2 - \dots\dots + (-1)^p\eta^p, \\ \Psi(\eta, \zeta) &= \beta_{p-1} - \beta_{p-2}\zeta + \beta_{p-3}\zeta^2 - \dots\dots + (-1)^{p-1}\zeta^{p-1}, \end{aligned}$$

where $\beta_k = \alpha_k - \alpha_{k-1}\eta + \alpha_{k-2}\eta^2 - \dots\dots + (-1)^k\eta^k$. Then

$$\Phi(\eta) = (\zeta - \eta)\Psi(\eta, \zeta) + \Phi(\zeta),$$

and it is clear that the result of the elimination of η between the equations

$$\Phi(\eta) = 0, \quad \Psi(\eta, \zeta) = 0,$$

is the equation

$$\Phi(\zeta) = 0.$$

Applying this remark to the successive elimination of .... ξ, ν, μ, λ between the equations (14), (10), (5), we readily see that the result of the elimination is the equation

$$F(\theta) = 0;$$

in other words, θ can be any one of the roots of the equation (5).

5. Let $\lambda_1, \lambda_2, \ldots\ldots, \lambda_m$ be the roots of (5), and let us, first, assume that no two of them are equal. To each value λ_k of θ corresponds a distinct integral $[w]_k$ of (13), containing n arbitrary constants. It can be shown that in this case the general solution of (1) is given by the formulas

$$y = \sum_{k=1}^{k=m} C_k [w]_k; \quad C_k = (-1)^{m-1} \frac{\partial \log \Delta}{\partial h_{k,\,m-1}};$$

$$\Delta = \begin{vmatrix} h_{1,0} & h_{1,1} & \ldots\ldots & h_{1,m-1} \\ h_{2,0} & h_{2,1} & \ldots\ldots & h_{2,m-1} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ h_{m,0} & h_{m,1} & \ldots\ldots & h_{m,m-1} \end{vmatrix}; \tag{15}$$

$$h_{k,\,e} = \lambda_k^e; \qquad k = 1, 2, \ldots\ldots, m; \; e = 0, 1, \ldots\ldots, m-1.$$

6. To show this, we recall the well-known relations in the theory of determinants

$$\sum_{k=1}^{k=m} C_k h_{k,\,e} = 0, \qquad e = 0, 1, \ldots\ldots, m-2;$$

$$\sum_{k=1}^{k=m} C_k h_{k,\,m-1} = (-1)^{m-1};$$

or, introducing the values of $h_{k,\,e}$,

$$\left.\begin{aligned}&\sum_{k=1}^{k=m} C_k\lambda_k^e = 0, \qquad e = 0,\, 1,\, \ldots\ldots,\, m-2\,;\\&\sum_{k=1}^{k=m} C_k\lambda_k^{m-1} = (-1)^{m-1}.\end{aligned}\right\} \tag{16}$$

The operation $D^{(n)}$ performed on the expression (15) of y gives

$$y_1 = \sum_{k=1}^{k=m} C_k\,[w_1]_k,$$

and, by (13), therefore,

$$y_1 = -\sum_{k=1}^{k=m} C_k\lambda_k\,[w]_k + f(x)\sum_{k=1}^{k=m} C_k.$$

But, by the first of the relations (16), viz. for $e = 0$, we have $\Sigma C_k = 0$, hence

$$y_1 = -\sum_{k=1}^{k=m} C_k\lambda_k\,[w]_k. \tag{17}$$

Likewise, the operation $D^{(n)}$ performed on (17) gives, by (13) and the relation (16) for $e = 1$,

$$y_2 = \sum_{k=1}^{k=m} C_k\lambda_k^2\,[w]_k.$$

Continuing the application of the symbol $D^{(n)}$ successively we find that

$$\begin{aligned}y_e &= (-1)^e\sum_{k=1}^{k=m} C_k\lambda_k^e\,[w]_k, \qquad e = 0,\, 1,\, \ldots\ldots,\, m-1\,;\\y_m &= (-1)^m\sum_{k=1}^{k=m} C_k\lambda_k^m\,[w]_k + f(x),\end{aligned}$$

from which it follows that

$$\begin{aligned}&y_m + a_1y_{m-1} + a_2y_{m-2} + \ldots\ldots + a_my\\&\qquad = \sum_{k=1}^{k=m} C_k\,[w]_k\,\{a_m - a_{m-1}\lambda_k + a_{m-2}\lambda_k^2 - \ldots\ldots + (-1)^m\lambda_k^m\} + f(x),\end{aligned}$$

and, λ_k being a root of (5), the right-hand side of the last equation reduces to $f(x)$. Q. E. D.

7. We pass now to the case of equal roots. Let us, for the present, assume that $\lambda_1 = \lambda_2 = \ldots\ldots = \lambda_i$, and that all the other roots of (5) are distinct. In this case the general solution of (1) is given by the formulas

$$y = \sum_{k=1}^{k=i} C_k \frac{d^{k-1}[w]_1}{d\lambda_1^{k-1}} + \sum_{k=1}^{k=m-} C'_{i+k}[w]_{i+k}; \tag{18}$$

where the arbitrary constants in $[w]_1$, $\frac{d[w]_1}{d\lambda_1}$, $\frac{d^2[w]_1}{d\lambda_1^2}$, ..., $\frac{d^{i-1}[w]_1}{d\lambda_1^{i-1}}$ are *independent of one another.* Thus, if

$$[w]_1 = \phi(x, \lambda_1, \alpha_1, \alpha_2, \ldots\ldots, \alpha_n),$$

we will understand under $\frac{d[w]_1}{d\lambda_1}$ the function

$$\frac{d\phi(x, \lambda_1, \alpha'_1, \alpha'_2, \ldots\ldots, \alpha'_n)}{d\lambda_1},$$

with arbitrary constants $\alpha'_1, \alpha'_2, \ldots\ldots, \alpha'_n$ independent of $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$.

As to the coefficients C_k and C'_{i+k}, they will be determined by the formulas

$$C_k = (-1)^{m-1}\frac{\partial \log \Delta}{\partial h_{k,\,m-1}}, \quad C'_{i+k} = (-1)^{m-1}\frac{\partial \log \Delta}{\partial h_{i+k,\,m-1}};$$

$$\Delta = \begin{vmatrix} h_{1,0} & h_{1,1} & \ldots\ldots & h_{1,m-1} \\ h_{2,0} & h_{2,1} & \ldots\ldots & h_{2,m-1} \\ \ldots & \ldots & \ldots\ldots & \ldots \\ \ldots & \ldots & \ldots\ldots & \ldots \\ h_{m,0} & h_{m,1} & \ldots\ldots & h_{m,m-1} \end{vmatrix}; \tag{19}$$

$$h_{k,\,e} = \frac{d^{k-1}\lambda_1^e}{d\lambda_1^{k-1}}, \qquad k = 1, 2, \ldots\ldots, i;$$

$$h_{i+k,\,e} = \lambda_{i+k}^e, \qquad k = 1, 2, \ldots\ldots, m-i, \qquad e = 0, 1, 2, \ldots\ldots, m-1.$$

Moreover, we will assume that $C_k = 0$ for $k > i$, and $C'_j = 0$ for $j < i + 1$.

8. To prove that (18) is the general solution of (1) in this case, we observe again that

$$\left.\begin{aligned} &\sum_{k=1}^{k=i} C_k h_{k,\,e} + \sum_{k=1}^{k=m-i} C'_{i+k} h_{i+k,\,e} = 0\,, \qquad e = 0,\ 1,\ \ldots\ldots,\ m-2\,; \\ &\sum_{k=1}^{k=i} C_k h_{k,\,m-1} + \sum_{k=1}^{k=m-i} C'_{i+k} h_{i+k,\,m-1} = (-1)^{m-1}. \end{aligned}\right\} \qquad (20)$$

We shall, besides, make use of the following relation which is based on the principle of interchangeability of the order of differentiation and on the equation (13):

$$D^{(n)} \frac{d^k [w]_1}{d\lambda_1^k} = \frac{d^k D^{(n)} [w]_1}{d\lambda_1^k} = \frac{d^k [w_1]_1}{d\lambda_1^k} = -\,\lambda_1 \frac{d^k [w]_1}{d\lambda_1^k} - k\,\frac{d^{k-1} [w]_1}{d\lambda_1^{k-1}}\,. \qquad (21)$$

9. The operation $D^{(n)}$ performed on the expression (18) of y gives, account being taken of the relation (21),

$$y_1 = -\sum_{k=1}^{k=i} (k C_{k+1} + \lambda_1 C_k)\, \frac{d^{k-1}[w]_1}{d\lambda_1^{k-1}} - \sum_{k=1}^{k=m-i} C'_{i+k} \lambda_{i+k} [w]_{i+k}$$

$$+ f(x) \left\{ C_1 + \sum_{k=1}^{k=m-i} C'_{+k} \right\}.$$

But by the first of the relations (20), viz. for $e = 0$, $C_1 + \Sigma C'_{i+k} = 0$; hence

$$y_1 = -\sum_{k=1}^{k=i} (k C_{k+1} + \lambda_1 C_k)\, \frac{d^{k-1}[w]_1}{d\lambda_1^{k-1}} - \sum_{k=1}^{k=m-i} C'_{i+k} \lambda_{i+k} [w]_{i+k}\,. \qquad (22)$$

Continuing the application of the symbol $D^{(n)}$ successively to $y_1, y_2, \ldots\ldots$, we find, in general,

$$y_e = (-1)^e \sum_{k=1}^{k=i} M_{k,\,e}\, \frac{d^{k-1}[w]_1}{d\lambda_1^{k-1}} + (-1)^e \sum_{k=1}^{k=m-i} C'_{i+k} \lambda_{i+k}^e [w]_{i+k}\,, \qquad (23)$$

where

$$
\begin{aligned}
(k-1)!\,M_{k,e} = (k+e-1)!\,C_{k+e} + \binom{e}{1}(k+e-2)!\,C_{k+e-1}\lambda_1 + \ldots\ldots \\
+ \binom{e}{j}(k+e-j-1)!\,C_{k+e-j}\lambda_1^j + \ldots\ldots \\
+ \binom{e}{1} k!\,C_{k+1}\lambda_1^{e-1} + (k-1)!\,C_k\lambda_1^e; \quad (24)
\end{aligned}
$$

$$e = 0, 1, 2, \ldots\ldots, m-1;$$

and

$$y_m = (-1)^m \sum_{k=1}^{k=m} M_{k,m} \frac{d^{k-1}[w]_1}{d\lambda_1^{k-1}} + (-1)^m \sum_{k=1}^{k,=m-i} C'_{i+k}\lambda_{i+k}^m [w]_{i+k} + f(x). \quad (25)$$

10. From the preceding formulas it follows that

$$
\begin{aligned}
y_m + a_1 y_{m-1} + a_2 y_{m-2} + \ldots\ldots + a_m y = f(x) \\
+ (-1)^m \sum_{k=1}^{k=i} \frac{d^{k-1}[w]_1}{d\lambda_1^{k-1}} \{M_{k,m} - a_1 M_{k,m-1} + a_2 M_{k,m-2} - \ldots\ldots + (-1)^m a_m M_{k,0}\} \\
+ (-1)^m \sum_{k=1}^{k=m-i} C'_{i+k}[w]_{i+k}\{\lambda_{i+k}^m - a_1\lambda_{i+k}^{m-1} + a_2\lambda_{i+k}^{m-2} - \ldots\ldots + (-1)^m a_m\}.
\end{aligned}
$$

But, λ_{i+k} being a root of (5), the last sum on the right-hand side of this equation vanishes, and to prove that (18) is the general solution of (1), we, therefore, only need to show that

$$M_{k,m} - a_1 M_{k,m-1} + a_2 M_{k,m-2} - \ldots\ldots + (-1)^m M_{k,0} = 0. \quad (26)$$

Now, it is easy to verify that the left-hand side of (26) may be presented in the form

$$\sum_{e=0}^{e=m} \frac{(k+e-1)!}{(k-1)!\,e!} C_{k+e} \frac{d^e F(\lambda_1)}{d\lambda_1^e}.$$

But, λ_1 being a multiple root of (5) of the order i, we have

$$\frac{d^e F(\lambda_1)}{d\lambda_1^e} = 0, \text{ for } e = 0, 1, 2, \ldots\ldots, i-1;$$

on the other hand, for $e \geqq i$, since k is at least equal to one, $C_{k+e} = 0$ (see 7). Therefore, the left-hand side of (26) vanishes for all the values of $k = 1, 2, \ldots, i$.
Q. E. D.

11. It can be readily seen now that, in the most general case, when (5) admits of several multiple roots $\lambda_1, \lambda_2, \lambda_3, \ldots$ of respective orders $\alpha_1, \alpha_2, \alpha_3, \ldots$, with the condition

$$\alpha_1 + \alpha_2 + \alpha_3 + \ldots = m,$$

the general solution of (1) will be given by the formulas

$$y = \sum_{k=1}^{k=\alpha_1} C'_k \frac{d^{k-1}[w]_1}{d\lambda_1^{k-1}} + \sum_{k=1}^{k=\alpha_2} C''_k \frac{d^{k-1}[w]_2}{d\lambda_2^{k-1}} + \sum_{k=1}^{k=\alpha_3} C'''_k \frac{d^{k-1}[w]_3}{d\lambda_3^{k-1}} + \ldots; \tag{27}$$

$$C'_k = (-1)^{m-1} \frac{\partial \log \Delta}{\partial h'_{k,\,m-1}}, \qquad k = 1, 2, \ldots, \alpha_1; \qquad h'_{k,\,e} = \frac{d^{k-1}\lambda_1^e}{d\lambda_1^{k-1}};$$

$$C''_k = (-1)^{m-1} \frac{\partial \log \Delta}{\partial h''_{k,\,m-1}}, \qquad k = 1, 2, \ldots, \alpha_2; \qquad h''_{k,\,e} = \frac{d^{k-1}\lambda_2^e}{d\lambda_2^{k-1}};$$

$$C'''_k = (-1)^{m-1} \frac{\partial \log \Delta}{\partial h'''_{k,\,m-1}}, \qquad k = 1, 2, \ldots, \alpha_3; \qquad h'''_{k,\,e} = \frac{d^{k-1}\lambda_3^e}{d\lambda_3^{k-1}};$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\Delta = \left| \begin{array}{llll} h'_{1,\,0} & h'_{1,\,1} & \ldots & h'_{1,\,m-1} \\ h'_{2,\,0} & h'_{2,\,1} & \ldots & h'_{2,\,m-1} \\ \cdots & \cdots & \cdots & \cdots \\ h'_{\alpha_1,\,0} & h'_{\alpha_1,\,1} & \ldots & h'_{\alpha_1,\,m-1} \\ h''_{1,\,0} & h''_{1,\,1} & \ldots & h''_{1,\,m-1} \\ \cdots & \cdots & \cdots & \cdots \\ h''_{\alpha_2,\,0} & h''_{\alpha_2,\,1} & \ldots & h''_{\alpha_2,\,m-1} \\ h'''_{1,\,0} & h'''_{1,\,1} & \ldots & h'''_{1,\,m-1} \\ \cdots & \cdots & \cdots & \cdots \\ h'''_{\alpha_3,\,0} & h'''_{\alpha_3,\,1} & \ldots & h'''_{\alpha_3,\,m-1} \\ \cdots & \cdots & \cdots & \cdots \end{array} \right|$$

12. An important case of the equation (1), considered in this paper, occurs in some problems of elasticity, namely, when $n = 2$ and

$$D^{(2)} \equiv \frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{p^2}{x^2}.$$

The problem is then reduced to the integration of a Bessel equation with second member

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\left(\frac{p^2}{x^2}-\lambda\right)y=f(x).$$

This special case has been discussed by the author in a paper before the Academy of Sciences of Paris.*

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* Sur une classe d'équations différentielles réductibles à l'équation de Bessel. C. R. 11 Mai, 1903. See also a paper by the author in C. R. 27 Octobre, 1902: Sur l'équation de Bessel avec second membre.